The Five Stages of Place Value

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I. The power of the Principles of Place Value.

Place value is perhaps the most undervalued aspect of mathematics in the elementary school curriculum. Justification of this claim should start by clarifying that what is meant here by “place value” is not what is most commonly understood by this term. Place value is normally considered as a vocabulary issue: in the number 234, the 2 is in the hundreds place, the 3 is in the tens place, and the 4 is in the ones place. However, the essential idea of place value is better represented by the statement:

Every number can be represented as a sum of numbers of a very special kind. \(\text{PV1}\)

Moreover, this expression is unique: exactly one representation for each number.\(^1\)

The very special numbers are numbers like 4 and 30 and 200 and 7,000, numbers whose base ten representation has only one non-zero digit. The usual way of writing a number is a convention for expressing it as a sum of the very special numbers. Thus, the symbol 234 should be understood as standing for a sum:

\[
234 = 200 + 30 + 4.
\]

Fine. Why is this useful? A key reason is is that it enables computation (i.e., addition and subtraction, multiplication and division). The fact that the basic arithmetic operations can be efficiently performed by effectively teachable algorithms was the reason that the place value system, which was only introduced into Europe in the late middle ages (around 1200), supplanted the well entrenched system of Roman numerals. Thus, a second key aspect of place value notation is:

Computations can be performed efficiently by \(\text{PV2a}\)

a) computing with pairs of the very special numbers, then

b) combining the results using the Rules of Arithmetic,\(^2\) (aka, Properties of the Operations).

In fact, it gets better: the precise nature of the very special numbers, in combination with the Rules of Arithmetic, imply that:

Computations with pairs of the very special numbers reduce to \(\text{PV2b}\)

a) single-digit computations, plus

b) order of magnitude considerations.

Thus, by just knowing the single-digit addition and multiplication facts, order of magnitude, and the Rules of Arithmetic, one can perform the operations of arithmetic on any whole numbers. Incredible power from such modest knowledge!

Explaining why computing with pairs of the very special numbers involves only the addition and multiplication facts requires knowing more about the nature of the very special numbers.

First we should know that each very special number is the product of a digit and a base ten unit, which is a very special number whose non-zero digit is a 1. Thus,

\[
4 = 4 \times 1 \quad 30 = 3 \times 10 \quad 200 = 2 \times 100 \quad 7,000 = 7 \times 1,000.
\]
This is enough to get us started with adding. We only try to add two very special numbers when their associated base ten units are the same. We will say that they have the same order of magnitude. Then all we have to do is add the digits. Thus, just as \( 2 + 3 = 5 \), we can say that

\[
20 + 30 = 50, \text{ and } 200 + 300 = 500, \text{ and } 2,000 + 3,000 = 5,000 \text{ and so forth.}
\]

To understand a little more: we need to know that the base ten units themselves have multiplicative structure. The smallest unit is of course 1, and the next one is 10. The choice of 10 for the first unit larger than 1 is just that, a choice, but all the larger units are determined by that choice: each base ten unit larger than 10 is just a product of a certain number of tens; that is, they are the numbers called the powers of 10.

Thus 100 = 10 x 10, 1,000 = 10 x 10 x 10, 10,000 = 10 x 10 x 10 x 10.

Very conveniently (an aspect of the refined subtlety of place value notation), the number of zeros in the standard representation for the unit tells you how many 10s are multiplied together to make the unit. From this, we can easily find the product of any two base ten units. The result is always another base ten unit! For example,

\[
10 \times 10 = 100 \quad 10 \times 100 = 1,000 \quad 100 \times 100 = 10,000 \quad 1,000 \times 1,000 = 1,000,000 \text{ and so forth.}
\]

The rule is, in multiplying two base ten units, the number of zeros in the product is the just the total number of zeros in the two factors.

From this, it is an easy matter to multiply any two very special numbers: you multiply the base ten units, and you multiply the digits. Thus, just as \( 2 \times 3 = 6 \), we have

\[
2 \times 30 = (2 \times 1) \times (3 \times 10) = (2 \times 3) \times (1 \times 10) = 6 \times 10 = 60
\]

\[
200 \times 300 = (2 \times 100) \times (3 \times 100) = (2 \times 3) \times (100 \times 100) = 6 \times 10,000 = 60,000 \text{ etc, etc.}
\]

Thus, the specific nature of the very special numbers means that the rules for adding or multiplying them are related in a simple way to the addition and multiplication facts for the digits. Then, as stated in (PV2), the Rules of Arithmetic allow computations with any numbers to be performed by appropriate combinations of computations with the very special numbers. The commonly used algorithms of arithmetic provide efficient ways of organizing the necessary steps. A detailed discussion of how (and why!) they work is given in many places, for example [1], [2] or [3]. The purpose of this note is simply to emphasize how the algorithms flow from the simple principles of place value, (PV1) and (PV2).

II. Teaching the Principles of Place Value.

The discussion above about the nature of the very special numbers can be summarized in a sequence of equalities:

\[
234 = 2 \times 100 + 3 \times 10 + 4 = 2 \times (10 \times 10) + 3 \times 10 + 4 = 2 \times 10^2 + 3 \times 10 + 4 = 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0.
\]

Each of these expressions represents a conceptual advance over the previous one. The second expression, which is often called expanded form, recognizes that base ten notation is shorthand for the decomposition of a number as a sum of the very special numbers. The third expression (which might be called the second expanded form) recognizes the first stage of the multiplicative structure of the very special numbers: that each is a digit times a base ten unit. The fourth expression (the third expanded form) recognizes the multiplicative structure of the base ten units: each is a repeated product of 10s (aka, a power of 10). Finally, the last expression (the polynomial form) summarizes the structure using the algebraic notation of exponents and powers. It expresses a base ten number as a “polynomial in 10”, and as such provides a natural
link to algebra. It also displays a “very special number” as being a digit times a power of 10. This final expression captures the reason the place value system is so powerful: it is harnessing all the operations of algebra, simply to write numbers. It may be helpful to put all this together in one mouthful: any whole number can be expressed as a sum of digital multiples of powers of 10.9

As noted, each of the five expressions in equation (PV3) represents a conceptual advance over the preceding one. Accordingly, we will call these five expressions the **five stages of place value understanding.** The elementary mathematics curriculum should be designed to lead students carefully through all these stages of understanding. Beginning with two-digit numbers in first grade, students should develop the habit of thinking of base ten numbers in terms of expanded form. First graders should become comfortable with the fact that a two digit number is made of some 10s and some 1s, and that they can add and subtract with the 10s and the 1s independently (perhaps finishing with some regrouping). This habit should be reinforced in later grades, and made the basis for understanding computation. Students should learn the second expanded form when they are introduced to multiplication (in 3rd grade). They should learn the third expanded form in 4th grade, as part of understanding the relationship between the base ten units (each is 10 times the next smaller one, and 1/10 times the next larger one)10, and as part of preparation for extending the base ten system from whole numbers to decimal fractions (grade 5). Finally, the polynomial form should be discussed and worked with when exponential notation is introduced (6th grade). The familiar procedures for multiplication should be related to the Law of Exponents, and especially, the extension to negative exponents (for decimal fractions as noted above) should be made explicit (8th grade). The last form should be reviewed and related to polynomial algebra when this is studied in high school.

### III. The Need for a Name.

The reader may be tired of the bland and non-descriptive phrase “very special number”. The authors like it even less. One of the most telling symptoms of the under appreciation of the principles of place value is that there is no standard, simple, descriptive name for these numbers in our curricula.11 Perhaps in an era when the main goal of instruction in arithmetic was to produce effective practitioners, there was no need to discuss the topic from a conceptual point of view, or to make explicit the many incarnations of expanded form. However, we are well beyond that era, and the goals of mathematics instruction are more ambitious now. In mathematics education, conceptual understanding of the more elementary topics is needed to support learning of the more advanced ones. Not having a good name for the key building blocks of the place value system impedes discussion of conceptual issues, and is a roadblock to teaching and learning of its conceptual structure. We need a good name.12

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1 This discussion is limited to whole numbers. We do not consider the complications involved in allowing infinite decimal expansions.

2 There are 9 Rules of Arithmetic: 4 for addition, 4 parallel ones for multiplication, and one (the Distributive Rule) that connects addition and multiplication. It is not necessary to know them in detail to appreciate our main argument. References to various Rules to justify technical claims have been put in footnotes, and can be ignored by the reader not familiar with them.

3 Lying behind this simplicity is the *Distributive Rule*.

4 Of course the choice of 10 as a base is a historical fact. We could in principle use any positive integer \( b \) as the base of our system, and then we would represent numbers as sums of multiples of powers of \( b \). See[4] for a discussion of place value when the base \( b \) is not 10.

5 Note that we haven’t put any parentheses in these products. This is OK because of the *Associative Rule for Multiplication*.

6 (and again using the Associative Rule for Multiplication)

7 Thanks now to the *Commutative Rule for Multiplication*, as well as the Associative Rule.

8 The rules above for sum and products of the very special numbers work just as well if the sum or product of the digits is ten or more. E.g., 70 + 50 = 120\( = 12 \times 10 \), and 70x50 = 3500\( = 35 \times 100 \),
just as \(7+5 = 12\) and \(7 \times 5 = 35\). However, the result will then not be a very special number, and may have to be decomposed into its very special components for further processing. This is known as \textit{regrouping} (aka, \textit{carrying}).

Notice that, even though we are discussing here only integers, this system smoothly extends to finite decimal fractions like \(17.32 = 1 \cdot 10^1 + 7 \cdot 10^0 + 3 \cdot 10^{-1} + 2 \cdot 10^{-2}\). Any real number can be approximated with arbitrary precision by decimal fractions.

This is also the key to using base ten numbers in estimation, which is very important, but not discussed here.

We have seen that a very special number can be described as a “digit times a power of 10”. However, this description is neither short nor simple.

In [3], they are called “single place numbers”. Because the process of “rounding numbers” involves replacing digits with zeros, perhaps “very round number” might be a reasonable name, although some may think it lacks dignity.

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References


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