Problem Corner

A Geometric EXPLORATION

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A question about angle bisectors
Consider a $\triangle ABC$ in which D, E and F are the midpoints of the sides BC, CA and AB respectively. Let G be the centroid of triangle ABC, i.e., the point of intersection of the medians AD, BE and CF. It is well-known that G is also the centroid of triangle DEF.

If, instead of being the midpoints, the points D, E and F are the points of intersection of the internal bisectors of $\measuredangle BAC$, $\measuredangle ABC$ and $\measuredangle ACB$ respectively with the opposite sides (BC, CA and AB respectively), then do the incentres of triangles ABC and DEF coincide? They do, if $\triangle ABC$ is equilateral. Are there triangles other than the equilateral triangle with such a property? Let us analyze.

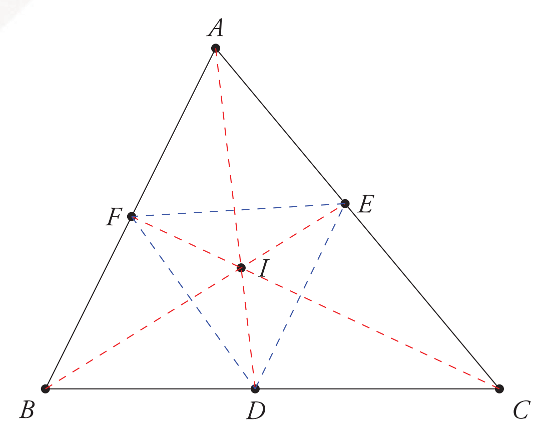

Figure 1.

Let I be the common incentre of $\triangle ABC$ and $\triangle DEF$. Observe that AD bisects both $\measuredangle BAC$ and $\measuredangle EDF$. In $\triangle AFD$ and $\triangle AED$, side AD is common, $\measuredangle DAF = \measuredangle DAE$ and $\measuredangle ADF = \measuredangle ADE$. Therefore, $\triangle AFD \cong \triangle AED$. Hence $DE = DF$ and $AE = AF$.

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By a similar argument, $\triangle BDE \cong \triangle BFE$, hence $DE = EF$ and $BD = BF$. Summarising, we obtain $DE = EF = DF$. Thus DEF is an equilateral triangle.

Also observe that $\triangle AFE$ and $\triangle BDF$ are isosceles. Thus, $\measuredangle AFE = 90° - \frac{1}{2}\measuredangle BAC$ and $\measuredangle BFD = 90° - \frac{1}{2}\measuredangle ABC$. But

$$\measuredangle AFE + \measuredangle EFD + \measuredangle BFD = 180°. \quad (1)$$

Therefore

$$90° - \frac{\measuredangle BAC}{2} + 60° + 90° - \frac{\measuredangle ABC}{2} = 180°, \quad (2)$$

whence $\measuredangle ACB = 60°$. Similarly, we can show that $\measuredangle ABC = 60°$ and $\measuredangle BAC = 60°$ and we see that $\triangle ABC$ must be equilateral.

Therefore there does not exist any triangle other than an equilateral triangle with such a property.

The same question about altitudes

What if D, E and F are the feet of the altitudes from A, B and C on to the sides BC, CA and AB respectively? When do the orthocentres of ABC and DEF coincide? In this case one has to do a more careful analysis.

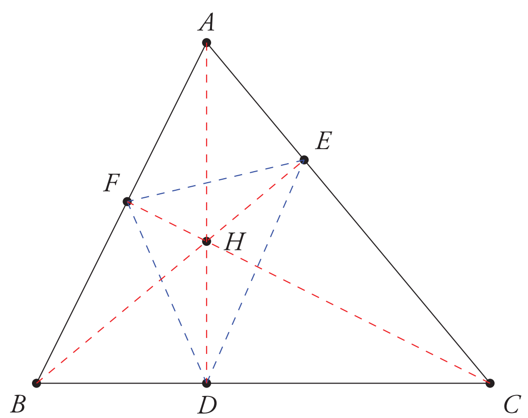

Figure 2.

If $\triangle ABC$ is acute-angled, then points D, E and F lie in the interior of line segments BC, CA and AB respectively. Thus the orthocentre (the point of intersection of the altitudes AD, BE and CF) of ABC, denoted by H, lies in the interior of the triangle.

Moreover, H is the incentre of $\triangle DEF$. To see this, it is enough to prove that AD, BE and CF internally bisect $\measuredangle FDE$, $\measuredangle DEF$ and $\measuredangle EFD$ respectively. Observe that in quadrilateral $BDHF$, $\measuredangle BFH = \measuredangle BDH = 90°$, i.e., a pair of opposite angles are supplementary. Therefore it is cyclic, hence $\measuredangle HDF = \measuredangle HBF = \measuredangle EBA = 90° - A$. Similarly, one may observe that quadrilateral $CDHE$ is cyclic and conclude that $\measuredangle HDE = \measuredangle HCE = \measuredangle FCA = 90° - A$. Hence

$$\measuredangle HDE = \measuredangle 90° - A = \measuredangle HDF, \quad (3)$$

which shows that AD bisects $\measuredangle FDE$. By mimicking this proof, we may prove that BE and CF bisect $\measuredangle DEF$ and $\measuredangle EFD$, respectively.

Now if H is also the orthocentre of $\triangle DEF$, then in this triangle the incentre and the orthocentre coincide. This implies that the internal bisectors of the angles are also the altitudes on the opposite sides, and this leads us to conclude that $\triangle DEF$ is equilateral. This shows that

$$180° - 2A = 180° - 2B = 180° - 2C = 60°, \quad (4)$$

whence $A = B = C = 60°$.

If $\triangle ABC$ is right-angled with, say, $\measuredangle BAC = 90°$, then the points E and F, the feet of the altitudes from B and C to the opposite sides, coincide with A and $\triangle DEF$ degenerates to the line segment AD. If ABC is obtuse-angled with, say, $\measuredangle BAC > 90°$, then the points E and F lie on CA and BA produced beyond A, and their point of intersection, the orthocentre H, lies outside ABC. It also lies outside $\triangle DEF$. Why?

Another class of problems

Now we explore a different class of problems. Given $\triangle ABC$ and a point P in its interior, we draw the lines AP, BP, CP. Suppose that AP intersects BC at D, BP intersects CA at E, and CP intersects AB at F. If $\triangle DEF$ is equilateral, then does it follow that $\triangle ABC$ is equilateral?

We consider this question for some special positions of P inside the triangle, namely, when P is either the centroid (G) or the orthocentre(H) or the incentre (I).

- First, let P be the centroid G. In this case, $\triangle DEF$ is similar to $\triangle ABC$ and its sides are half as long as the sides of $\triangle ABC$. Thus, if $\triangle DEF$ is equilateral, then so is $\triangle ABC$.
- If P is the orthocentre H of $\triangle ABC$, then as we have assumed that P (or H) is an interior point, $\triangle ABC$ is acute-angled and the angles of $\triangle DEF$, as we have seen earlier, are $180^\circ - 2A$, $180^\circ - 2B$ and $180^\circ - 2C$. If each of these is 60°, then it readily follows that each of the angles $\measuredangle BAC$, $\measuredangle ABC$ and $\measuredangle ACB$ is 60°.
- When P is the incentre I of $\triangle ABC$, we claim that if $\triangle DEF$ is equilateral, then so is $\triangle ABC$. Can the reader prove this?

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